Topic :- 2nd order Equations with variable co-efficients (Removing the First Derivatives)

Class - B.Sc (Part II) Date -

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1. Solve $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \left(1+\dfrac{2}{x^2}\right)y = 0$

Here $P = -\dfrac{2}{x}$ $Q = 1+\dfrac{2}{x^2}$

$$\therefore \; v = e^{-\frac{1}{2}\int P\,dx}$$

$$= e^{-\frac{1}{2}\int -\frac{2}{x}dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= \left(1+\frac{2}{x^2}\right) - \frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{1}{4}\times\frac{4}{x^2}$$

$$= 1 + \frac{2}{x^2} - \frac{1}{x^2} - \frac{1}{x^2} = 1$$

$\therefore$ The Transformed equation is

$$\frac{d^2u}{dx^2} + u = 0$$

$$\Rightarrow \; u = A\cos x + B\sin x$$

Hence $y = vu = x(A\cos x + B\sin x)$

2.

Solve

$$x^2 \frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$$

<u>Solution</u> Here Given Equation

$$x^2 \frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\left(1+\frac{1}{x}\right)\frac{dy}{dx} + \left(1+\frac{2}{x}+\frac{2}{x^2}\right)y = 0$$

This the Given Equation is reduced to the standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Here $P = -2\left(1+\frac{1}{x}\right)$, $Q = 1+\frac{2}{x}+\frac{2}{x^2}$

and $R = 0$

In order to remove the First derivative we choose

$$e = e^{-\frac{1}{2}\int P\,dx}$$

$$= e^{-\frac{1}{2}\int -2\left(1+\frac{1}{x}\right)dx}$$

$$= e^{\left(1+\frac{1}{x}\right)dx}$$

$$= e^{x+\log x} = e^{x}\cdot e^{\log x}$$

$$= x e^{x}$$

Putting $y = e\,u$, the above equation reduces to

$$\frac{d^2u}{dx^2} + Q_1 u = 0 \quad (\because R_1 = 0)$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= 1 + \frac{2}{x} + \frac{2}{x^2} - \frac{1}{2}(-2)\left(0 - \frac{1}{x^2}\right) - \frac{1}{4} \times 4\left(1 + \frac{1}{x}\right)^2$$

$$= \frac{x^2 + 2x + 2}{x^2} - \frac{1}{x^2} - \left(1 + \frac{2}{x} + \frac{1}{x^2}\right)$$

$$= \frac{x^2 + 2x + 2}{x^2} - \frac{1}{x^2} - 1 - \frac{2}{x} - \frac{1}{x^2}$$

$$= \cancel{1} + \cancel{\frac{2}{x}} + \cancel{\frac{2}{x^2}} - \cancel{\frac{1}{x^2}} - \cancel{1} - \cancel{\frac{2}{x}} - \cancel{\frac{1}{x^2}}$$

$$= 0$$

$$\therefore \frac{d^2u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = c_1$$

$$\Rightarrow u = c_1 x + c_2$$

Hence $y = e\,u = x e^{x}(c_1 x + c_2)$

Ex Solve

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = \sec x\, e^{x}$$

Solution :-> The above equation is in standard form.

Here $P = -2\tan x$ and $R = \sec x \, e^{x}$

$Q = 5$

$$\rho = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-2\tan x)\,dx}$$

$$= e^{\int \tan x\,dx}$$

$$= e^{\log \sec x}$$

$$= \sec x$$

Putting $y = \rho u$ the Given Equation is reduced to

$$\frac{d^2u}{dx^2} + Q_1 u = R_1$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= 5 - \frac{1}{2}(-2\sec^2 x) - \frac{1}{4} \times 4\tan^2 x$$

$$= 5 + \sec^2 x - \tan^2 x$$

$$= 5 + 1$$

$$= 6$$

And $R_1 = \frac{R}{\rho} = \frac{\sec x \cdot e^{x}}{\sec x} = e^{x}$

Thus $\frac{d^2u}{dx^2} + 6u = e^x$

The Auxiliary Equation is

$$D^2 + 6 = 0$$

$$D^2 = -6$$

$$\Rightarrow \quad D = \pm\sqrt{6}\, i$$

$\therefore$ C.F $= C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x$

Also P.I $= \frac{e^x}{D^2+6} = \frac{e^x}{7}$

Therefore the generally solution is

$$u = C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x + \frac{e^x}{7}$$

$$\therefore \quad y = e u = \sec x \left[C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x\right] + \frac{e^x \sec x}{7}$$